Exact solutions to the modified Korteweg-de Vries equation.

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Abstract

A formula for certain exact solutions to the modified Korteweg-de Vries (mKdV) equation is obtained via the inverse scattering transform method. The kernel of the relevant Marchenko integral equation is written with the help of matrix exponentials as

$$\Omega(x+y;t) = Ce^{-(x+y)A}e^{8A^3t}B,$$

where the real matrix triplet (A, B, C) consists of a constant $p \times p$ matrix A with eigenvalues having positive real parts, a constant $p \times 1$ matrix B, and a constant $1 \times p$ matrix C for a positive integer p. Using separation of variables, the Marchenko integral equation is explicitly solved yielding exact solutions to the mKdV equation. These solutions are constructed in terms of the unique solution P to the Sylvester equation $AP + PA = BC$ or in terms of the unique solutions Q and N to the respective Lyapunov equations $A^\dagger Q + QA = C^\dagger C$ and $AN + NA^\dagger = BB^\dagger$, where the † denotes the matrix conjugate transpose. Two interesting examples are provided.

1 Introduction

Consider the focusing modified Korteweg-de Vries (mKdV) equation

$$u_t + u_{xxx} + 6|u|^2 u_x = 0\,, \tag{1.1}$$

where the subscripts denote the appropriate partial derivates, u represents a real scalar function and $(x,\, t) \in \mathbb{R}^2$.

The modified Korteweg-de Vries (mKdV) equation arises in applications to the dynamics of thin elastic rods [19], phonons in anharmonic lattices [22], meandering ocean jets [23], traffic congestion [15, 20, 17, 13], hyperbolic surfaces [24], ion acoustic solitons [18], Alfvén waves in collisionless plasmas [14], slag-metallic bath interfaces [4], and Schottky barrier transmission lines [27].

In this paper we present a method to construct certain exact solutions to (1.1) that are globally analytic on the entire xt-plane and decay exponentially

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as $x \to \pm\infty$ for each fixed $t \in \mathbb{R}$. The method used to obtain these solutions is based on the Inverse Scattering Transform (IST) [1]-[3], [16] and [21]. The IST matches eq.(1.1) to the first order system of ordinary differential equations

$$\frac{d\xi}{dx} = -i\lambda\xi + u(x,t)\,\eta, \tag{1.2a}$$

$$\frac{d\eta}{dx} = i\lambda\eta - u(x,t)\,\xi, \tag{1.2b}$$

known as the Zakharov-Shabat system [26]. In Section 2 we develop its direct and the inverse scattering theory. More precisely, representing the corresponding scattering data in terms of a matrix realization [5]-[7], the Marchenko integral equation is separable and hence, can be solved algebraically. Its solution is easily related to the solution of (1.1).

The method used in this article has several advantages:

1. It is generalizable to the matrix version of eq.(1.1) and to other (matrix) nonlinear evolution equations (see, e.g., [11]).
2. The explicit formulas found in this paper are expressed in a concise form in terms of a triplet (A, B, C) where A is a real square matrix of dimension p, C is a real row vector and B is a real column vector. When the matrix order of A is very large, we have an explicit formula for the solution of the initial value problem for (1.1). Using computer algebra, we can "unzip" the solution in terms of exponential, trigonometric, and polynomial functions of x and t, but this unzipped expression may take several pages!
3. Our method easily treats nonsimple bound-state poles and the time evolution of the corresponding bound-state norming constants.

This paper is organized as follows: In Section 2 we develop the direct and the inverse scattering theory for system (1.2) and describe how the IST allows us to get the solution to (1.1). In Section 3 we construct the exact solutions to the initial value problem (1.1) in terms of real matrices (A, B, C) solving the Marchenko equation. In Section 4 we write the triplet (A, B, C) in a "canonical" form. Finally, in Section 5 we discuss the one-soliton and a multipole solution as examples.

2 IST method for the mKdV equation

The mKdV equation (1.1) is solvable by the inverse scattering tranform (IST). This method associates (1.1) with the first order system of ODE (1.2) where the coefficient u is called the potential. We suppose that u is a real scalar function belonging to $L^1(\mathbb{R})$. In this section we recall the basic ideas behind the IST and introduce the scattering coefficients and Marchenko integral equations. A more detailed exposition can be found in [1]-[3], [16] and [21].

Let us introduce the *Jost functions* from the right $\overline{\psi}(\lambda, x)$ and $\psi(\lambda, x)$, the Jost functions from the left $\phi(\lambda, x)$ and $\overline{\phi}(\lambda, x)$, and the Jost matrix solutions

$\Psi(\lambda, x)$ and $\Phi(\lambda, x)$ from the right and from the left as those solutions of (1.2) satysfying the asymptotic conditions:

$$\Psi(\lambda, x) = \begin{pmatrix}\overline{\psi}(\lambda, x) & \psi(\lambda, x)\end{pmatrix} = \begin{cases} e^{-i\lambda Jx}[I_2 + 0(1)], & x \to +\infty, \\ e^{-i\lambda Jx} a_l(\lambda) + o(1), & x \to -\infty, \end{cases}$$

$$\Phi(\lambda, x) = \begin{pmatrix}\phi(\lambda, x) & \overline{\phi}(\lambda, x)\end{pmatrix} = \begin{cases} e^{-i\lambda Jx}[I_2 + o(1)], & x \to -\infty, \\ e^{-i\lambda Jx} a_r(\lambda) + o(1), & x \to +\infty, \end{cases}$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Here we have not written the variable t. Since this system is first order, we have

$$\Phi(\lambda, x) = \Psi(\lambda, x) a_r(\lambda), \qquad \Psi(\lambda, x) = \Phi(\lambda, x) a_l(\lambda),$$

where $a_r(\lambda)$ and $a_l(\lambda)$ are called *transmission coefficient matrices*. It is easy to prove that $a_r(\lambda)$ and $a_l(\lambda)$, for all $\lambda \in \mathbb{R}$, are unitary matrices with unit determinant, one being the inverse of the other (see, e.g., [10, 3, 25]). It is convenient to write these matrices in the form

$$a_l(\lambda) = \begin{pmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda) \end{pmatrix}, \qquad a_r(\lambda) = \begin{pmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{pmatrix},$$

where $a_{lk}(\lambda)$ and $a_{rk}(\lambda)$ for $k = 1, 2, 3, 4$ are real scalar functions. The Jost functions $\phi(\lambda, x)$ and $\psi(\lambda, x)$ [10, 3, 25] are, for each $x \in \mathbb{R}$, analytic in $\mathbb{C}^+$, continuous in $\lambda \in \overline{\mathbb{C}^+}$ and tend to 1 as $|\lambda| \to \infty$ from within $\mathbb{C}^+$, where we use $\mathbb{C}^+$ ($\mathbb{C}^-$) to denote the upper (lower) complex open half planes, and $\overline{\mathbb{C}^\pm} = \mathbb{C}^\pm \cup \mathbb{R}$. Instead, the Jost functions $\overline{\phi}(\lambda, x)$ and $\overline{\psi}(\lambda, x)$ are, for each $x \in \mathbb{R}$, analytic in $\mathbb{C}^-$, continuous in $\lambda \in \overline{\mathbb{C}^-}$ and tend to 1 as $|\lambda| \to \infty$ from within $\mathbb{C}^-$. So, it is natural to consider the 2×2 matrices of functions

$$F_+(\lambda, x) = \begin{pmatrix}\phi(\lambda, x) & \psi(\lambda, x)\end{pmatrix}, \qquad F_-(\lambda, x) = \begin{pmatrix}\overline{\psi}(\lambda, x) & \overline{\phi}(\lambda, x)\end{pmatrix}.$$

As a result, $F_+(\lambda, x)$ is, for each $x \in \mathbb{R}$, analytic in $\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}$. Instead, $F_-(\lambda, x)$ is, for each $x \in \mathbb{R}$, analytic in $\mathbb{C}^-$ and continuous in $\overline{\mathbb{C}^-}$. Using this information we arrive at the Riemann-Hilbert problem

$$F_-(\lambda, x) = F_+(\lambda, x) J S(\lambda) J, \tag{2.1}$$

where

$$S(\lambda) = \begin{pmatrix} T_r(\lambda) & L(\lambda) \\ R(\lambda) & T_l(\lambda) \end{pmatrix}$$

is called the *scattering matrix*.

The direct problem can be formulated as follows: Starting from the potential $u(x)$, construct the scattering matrix or, equivalently, determine one reflection coefficient $R(\lambda)$ or $L(\lambda)$, the poles (bound-states) λ_j of the transmission coefficient $T_r(\lambda)$ or $T_l(\lambda)$ (see Theorem 3.16 of [10]), and the norming constants c_{js}

corresponding to those poles. Moreover, under the technical assumption that $a_{l1}(\lambda)$, $a_{r1}(\lambda)$, $a_{l4}(\lambda)$ and $a_{r4}(\lambda)$ are invertible for each $\lambda \in \mathbb{R}$, we can relate the transmission coefficients to the coefficients appearing in the scattering matrix $S(\lambda)$. More precisely, we have

$$\begin{aligned}
T_r(\lambda) &= a_{r1}(\lambda)^{-1}, & R(\lambda) &= -a_{l4}(\lambda)^{-1}a_{l3}(\lambda) = a_{r3}(\lambda)a_{r1}(\lambda)^{-1}, \\
T_l(\lambda) &= a_{l4}(\lambda)^{-1}, & L(\lambda) &= -a_{r1}(\lambda)^{-1}a_{r2}(\lambda) = a_{l2}(\lambda)a_{l4}(\lambda)^{-1}.
\end{aligned}$$

The inverse scattering problem consists of the (re)-construction of the unique potential $u(x)$ if one knows the scattering data. In this paper, following [5, 6], we solve this problem via the Marchenko method (see [21, 26]) as follows:

a. From the scattering data $\left\{R(\lambda),\, \left\{\lambda_j,\, \{c_{js}\}_{s=0}^{n_j-1}\right\}_{j=m+1}^{m+n}\right\}$, construct the function

$$\Omega(y) \stackrel{\text{def}}{=} \hat{R}(y) + \sum_{j=m+1}^{m+n} \sum_{s=0}^{n_j-1} c_{js} \frac{y^s}{s!} e^{i\lambda_j y} \tag{2.2}$$

where $\hat{R}(y) = \frac{1}{2\pi}\int_0^\infty R(\lambda)e^{i\lambda y}d\lambda$ represents the Fourier transform of $R(\lambda)$.

b. Solve the Marchenko integral equation

$$K(x,y) - \Omega(x+y)^\dagger + \int_x^\infty dz \int_x^\infty ds\, K(x,z)\,\Omega(z+s)\,\Omega(s+y)^\dagger = 0, \tag{2.3}$$

where $y > x$.

c. Construct the potential $u(x)$ by using the following formula:

$$u(x) = -2K(x,x). \tag{2.4}$$

Having presented the direct and inverse scattering problems corresponding to the LODE associated to the mKdV equation, we now discuss how the IST allows us to obtain the solution to the initial value problem for (1.1). The following diagram illustrates the IST procedure:

$$\begin{tikzcd}
\boxed{\begin{matrix}\text{given}\\ u(x,0)\end{matrix}} \arrow[r, "\substack{\text{direct scattering problem}\\ \text{with potential } u(x,0)}"] \arrow[d, "\substack{\text{mKdV}\\ \text{solution}}"] & \boxed{R(\lambda),\, \{\lambda_j,\, \{c_{js}(0)\}_{s=0}^{n_j-1}\}_{j=m+1}^{m+n}} \arrow[d, "\substack{\text{time evolution of}\\ \text{scattering data}}"'] \\
\boxed{u(x,t)} & \boxed{R(\lambda,t),\, \{\lambda_j,\, \{c_{js}(t)\}_{s=0}^{n_j-1}\}_{j=m+1}^{m+n}} \arrow[l, "\substack{\text{inverse scattering problem}\\ \text{with time evolved scattering data}}"]
\end{tikzcd}$$

To solve the initial value problem for (1.1), we use the initial condition $u(x,0)$ as a potential in the system (1.2). After that we develop the direct scattering theory as shown above and build the scattering matrix. Successively, let the initial scattering data evolve in time. The transmission coefficient does not change

in time and, as a consequence, also the bound states do not. The reflection coefficient $R(\lambda)$ evolves according to $R(\lambda,t)=e^{8\lambda^3 t}R(\lambda)$. It remains to determine the evolution of the norming constants. Extending our previous results on the nonlinear Schrödinger equation ([6, 11, 10, 9]) to the mKdV equation, we obtain the following time evolution of the norming constants:

$$\begin{pmatrix} c_{jn_j-1}(t) & \dots & c_{j0}(t) \end{pmatrix} = \begin{pmatrix} c_{jn_j-1}(0) & \dots & c_{j0}(0) \end{pmatrix} \, e^{-A_j^3 t},$$

where A_j is the matrix defined by eq. (4.7). Finally, we solve the inverse scattering problem by the Marchenko method for (1.2) with the time evolved scattering data $\left\{R(\lambda,t),\, \left\{\lambda_j,\, \{c_{js}(t)\}_{s=0}^{n_j-1}\right\}_{j=m+1}^{m+n}\right\}$, replacing in (2.2) $R(\lambda)$ with $R(\lambda,t)$ and c_{js} with $c_{js}(t)$. Then the function

$$u(x,t)=-2K(x,x;t)\,, \tag{2.5}$$

is a solution to the mKdV equation.

3 Explicit solutions for the mKdV equation

In this section we obtain two different but equivalent formulas yielding solutions to the mKdV equation. Using the expressions of the evolved reflection coefficient and the evolved norming constants in (2.2), we get

$$\Omega(y;t)=\frac{1}{2\pi}\int_0^\infty R(\lambda)e^{8\lambda^3 t}e^{i\lambda y}d\lambda+\sum_{j=m+1}^{m+n}\sum_{s=0}^{n_j-1}c_{js}(t)\frac{y^s}{s!}e^{i\lambda_j y}\,, \tag{3.1}$$

which satisfies the first order PDE

$$\Omega_t(y;t)+8\Omega_{yyy}(y;t)=0\,. \tag{3.2}$$

Now, following the approach of [5, 6, 7], we write the kernel $\Omega(y)$ introduced in (2.2) in the form

$$\Omega(y)=Ce^{-Ay}B, \quad y\geq 0\,, \tag{3.3}$$

where A, B, C are real matrices of size $p\times p$, $p\times 1$ and $1\times p$, respectively, for some integer p. Eq. (3.2) suggests us to take $\Omega(y;t)$ as

$$\Omega(y;t)=Ce^{-Ay}e^{8A^3 t}B,\ y\geq 0\,. \tag{3.4}$$

For reasons to be clarified later, we have some further requirements on the triplet $(A,\, B,\, C)$. More precisely,

1. Our triplets $(A,\, B,\, C)$ is a *minimal representation* of the kernel $\Omega(y)$, i.e,

   $$\bigcap_{r=1}^{+\infty}\ker CA^{r-1}=\bigcap_{r=1}^{+\infty}\ker B^\dagger(A^\dagger)^{r-1}=\{0\},$$

   and we refer the reader to [8] for many details on this subject.

2. All of the eigenvalues of the matrix A have positive real parts.

Following the procedure described in the preceding section, we find explicit solutions of the mKdV equation solving the Marchenko integral equation (2.3), where the kernel is given by (3.4) and the unknown function K depends on t. It is immediate to calculate

$$\Omega(y;t)^\dagger = B^\dagger e^{-A^\dagger y} e^{8(A^\dagger)^3 t} C^\dagger,\ y \ge 0\,. \tag{3.5}$$

By using (3.4) and (3.5), eq. (2.3) becomes

$$K(x,y;t) - \Big(B^\dagger e^{-A^\dagger x} - \int_x^\infty dz \int_x^\infty ds\, K(x,z;t)\, Ce^{-Az+8A^3t} e^{-As}\, BB^\dagger e^{-A^\dagger s}\Big) \\ \cdot\, e^{-A^\dagger y+8(A^\dagger)^3 t} C^\dagger = 0\,, \quad y > x\,. \tag{3.6}$$

Looking for a solution of (3.6) in the form

$$K(x,y;t) = H(x,t) e^{-A^\dagger y+8(A^\dagger)^3 t} C^\dagger\,, \tag{3.7}$$

and introducing the matrices Q and N as

$$Q = \int_0^\infty ds\, e^{-A^\dagger s} C^\dagger C e^{-As}, \qquad N = \int_0^\infty dr\, e^{-Ar} BB^\dagger e^{-A^\dagger r}\,, \tag{3.8}$$

after some easy calculations we obtain

$$H(x,t)\Gamma(x,t) = B^\dagger e^{-A^\dagger x}\,, \tag{3.9}$$

where (denoting by I_p the identity matrix of order p)

$$\Gamma(x,t) = I_p + e^{-A^\dagger x+8(A^\dagger)^3 t} Q e^{-2Ax+8A^3 t} N e^{-A^\dagger x}\,. \tag{3.10}$$

Substituting (3.9) into (3.7) we get

$$K(x,y;t) = B^\dagger e^{-A^\dagger x} \Gamma(x,t)^{-1} e^{-A^\dagger y+8(A^\dagger)^3 t} C^\dagger = B^\dagger F^{-1}(x,t) e^{-A^\dagger (y-x)} C^\dagger\,.$$

Putting

$$F(x,t) = e^{2A^\dagger x-8(A^\dagger)^3 t} + Q e^{-2Ax+8A^3 t} N\,, \tag{3.11}$$

we arrive at the solution formula

$$u(x,t) = -2B^\dagger F^{-1}(x,t) C^\dagger\,. \tag{3.12}$$

The matrices Q, N, $F(x,\,t)$ and the scalar function $u(x,\,t)$ introduced by (3.8), (3.11) and (3.12), respectively, satisfy the properties stated in the following

Theorem 3.1 *Suppose that the triplet $(A,\,B,\,C)$ is real and is a minimal representation of the kernel $\Omega(y;t)$, and that the eigenvalues of A have positive real parts. Then*

a) The matrices Q and N are real and positive selfadjoint, i.e $Q^\dagger = Q$ and $N^\dagger = N$.

b) The matrices Q and N are simultaneously invertible.

c) The matrix $F(x,t)$ is invertible on the entire xt-plane. Moreover, for each fixed t, $F(x,t)^{-1} \to 0$ as $x \to \pm\infty$.

d) The real scalar function $u(x,t)$ satisfies (1.1) *everywhere on the xt-plane. Moreover, $u(x,t)$ is analytic on the entire xt-plane and decays exponentially for each fixed t as $x \to \pm\infty$.*

Proof. The proof of the items *a*), *b*), *c*) and of the analyticity and asymptotic behaviour of the solution $u(x, t)$ is identical to the proof of items (ii) and (iii) of Therorem 4.2 of [7] (taking into account also Theorem 4.3 of the same paper). So we can refer the reader to this paper for details. However, we can verify directly that our solution (3.12) satisfies eq. (1.1). In order to do so, we use a slightly different notation. In particular, let us write formula (3.12) as

$$u(x;t) = -2B^\dagger e^{-A^\dagger x}\Gamma^{-1}(x,t)e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger \,, \tag{3.13}$$

where $\Gamma(x,\, t) = I_p + Q(x,\, t)N(x)$ with $Q(x,\, t) = e^{-A^\dagger x}e^{8(A^\dagger)^3 t}Qe^{-Ax}e^{8A^3 t}$ and $N(x) = e^{-Ax}Ne^{-A^\dagger x}$, Q and N being defined in (3.8). We will see in Section 4 that these two matrices are the unique solutions of the so-called Lyapunov equations, i.e. eqs. (4.1a). We recall the following rule: If $A(x)$ is an invertible matrix of functions depending on x such that its derivative with respect to x exists, then

$$\frac{\partial}{\partial x}\left(A(x)^{-1}\right) = -A(x)^{-1}\left(\frac{\partial}{\partial x}A(x)\right)A(x)^{-1}. \tag{3.14}$$

Applying this differentiation rule to the function $\Gamma(x,\, t)$ and deriving (3.13) with respect to t , we easily get (from now on we omit the dependence of $\Gamma(x,\, t)$ on x and t)

$$u_t = -16B^\dagger e^{-A^\dagger x}\Gamma^{-1}\left[(A^\dagger)^3 - Q(x,\, t)A^3N(x)\right]\Gamma^{-1}e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger \,. \tag{3.15}$$

Now, if one applies again (3.14) to the function Γ and take the x-derivative, after some straightforward calculations and using also (4.1a), we obtain

$$\left(\Gamma^{-1}\right)_x = \Gamma^{-1}A^\dagger + A^\dagger\Gamma^{-1} - 2\Gamma^{-1}(A^\dagger - QAN)\Gamma^{-1} \,. \tag{3.16}$$

As a consequence of (3.16), we can calculate in a direct way u_x obtaining

$$u_x = 4B^\dagger e^{-A^\dagger x}\Gamma^{-1}\left[(A^\dagger) - Q(x,\, t)AN(x)\right]\Gamma^{-1}e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger \,. \tag{3.17}$$

Now, computing the derivative of (3.17) and taking into account (3.16) we find

$$\begin{aligned} u_{xx} = 8B^\dagger e^{-A^\dagger x}\Gamma^{-1}&\left[(A^\dagger)^2 + QA^2N - 2(A^\dagger - QAN)\Gamma^{-1}(A^\dagger - QAN)\right] \\ &\cdot \Gamma^{-1}e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger \,. \end{aligned}$$

By very similar calculations we get

$$
\begin{aligned}
u_{xxx} &= 16B^\dagger e^{-A^\dagger x}\Gamma^{-1}\Big[(A^\dagger)^3 - QA^3N - 3((A^\dagger)^2 + QA^2N)\Gamma^{-1}(A^\dagger - QAN) \\
&- 3(A^\dagger - QAN)\Gamma^{-1}((A^\dagger)^2 + QA^2N) \\
&+ 6(A^\dagger - QAN)\Gamma^{-1}(A^\dagger - QAN)\Gamma^{-1}(A^\dagger - QAN)\Big]\Gamma^{-1}e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger\,.
\end{aligned}
\tag{3.18}
$$

Finally, using (3.17) and taking into account that u is a real scalar function, we obtain

$$
\begin{aligned}
&2|u|^2 u_x = u_x u^\dagger u + u u^\dagger u_x = 16B^\dagger e^{-A^\dagger x}\Gamma^{-1} \\
&\cdot \Big[((A^\dagger)^2 + QA^2N)\Gamma^{-1}(A^\dagger - QAN) + (A^\dagger - QAN)\Gamma^{-1}((A^\dagger)^2 + QA^2N) \\
&- 2(A^\dagger - QAN)\Gamma^{-1}(A^\dagger - QAN)\Gamma^{-1}(A^\dagger - QAN)\Big]\Gamma^{-1}e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger\,,
\end{aligned}
\tag{3.19}
$$

and, at this point, it is very simple to observe that $u_{xxx} + 6|u|^2 u_x = -u_t$. ■

Now, we will build a different explicit formula which is equivalent to the one expressed by (3.12). In order to obtain this result, we first observe that $\Omega(y,\, t)$, as a consequence of the realness of the triplet $(A,\, B,\, C)$, is a real function. As a result,

$$\Omega(y;\, t)^\dagger = \Omega(y;\, t).$$

By using this relation, eq. (2.3) can be written as

$$
\begin{aligned}
K(x,y;t) - \Big(Ce^{-Ax} - \int_x^\infty dz \int_x^\infty ds\, K(x,z;t)\, Ce^{-Az+8A^3t}e^{-As}\, BCe^{-A^\dagger s}\Big) \\
\cdot\, e^{-Ay+8(A)^3t}B = 0\,, \quad y > x\,.
\end{aligned}
\tag{3.20}
$$

By very similar computations we get the solution of (3.20) as

$$K(x,y;t) = CE^{-1}(x,t)e^{-A^\dagger(y-x)}B\,,$$

where

$$E(x,\, t) = e^{2Ax-8A^3t} + Pe^{-2Ax+8A^3t}P, \quad P = \int_0^\infty ds\, e^{-As}BCe^{-As}\,. \tag{3.21}$$

Finally, using eq. (2.5) we obtain

$$v(x;t) = -2CE^{-1}(x,t)B\,. \tag{3.22}$$

The next theorem shows the relationships between the matrices Q, N and P and the solutions formula (3.12) and (3.22).

Theorem 3.2 *Suppose that the triplet* (A, B, C) *is real and is a minimal representation of the kernel* $\Omega(y,t)$*, and that the eigenvalues of the matrix* A *have positive real parts. Then*

i. *The following relation holds:* $NQ = P^2$.

ii. *The matrix* P *is invertible on the entire* xt*-plane.*

iii. $E(x, t) = F(x, t)^\dagger$ *and, as a consequence, the matrix* $E(x, t)$ *is invertible on the entire* xt*-plane.*

iv. *For each fixed* t, $E(x, t)^{-1} \to 0$ *as* $x \to \pm\infty$.

v. *The real scalar function* $v(x,t)$ *satisfy* (1.1) *everywhere on the* xt*-plane. Moreover,* $v(x,t)$ *is analytic on the entire* xt*-plane and tends to zero exponentially for each fixed* t *as* $x \to \pm\infty$.

vi. *The explicit formulas* (3.12) *and* (3.22) *yield equivalent exact solutions to the mKdV eq.* (1.1) *everywhere on the entire* xt*-plane.*

We skip the proof of this theorem, because it can be constructed by rearranging (with inessential modifications) the proofs of Theorems 5.2, 5.4 and 5.5 of [7]. However, we remark that:

- By very similar calculations to those developed in the proof of Theorem 3.1, we can directly verify that the formula (3.22) satisfies eq. (1.1) everywhere on the xt-plane.
- Because $u(x, t)$ is real and scalar, the equivalence of (3.12) and (3.22) follows from the relation $E(x, t) = F(x, t)^\dagger$.

We conclude this section with the following

Theorem 3.3 *Suppose that the triplet* (A, B, C) *is real and is a minimal representation of the kernel* $\Omega(y,t)$*, and that the eigenvalues of the matrix* A *have positive real parts. Then the solution to the mKdV equation given in the equivalent forms* (3.12) *and* (3.22) *satisfies*

$$[u_x(x,t)]^2 = \frac{\partial^2 \log(\det E(x,t))}{\partial x^2} = \frac{\partial^2 \log(\det F(x,t))}{\partial x^2}. \tag{3.23}$$

The proof of this theorem can be found in [7, 6].

4 Canonical form of the triplet (A, B, C).

In this section we show how it is possible, without loss of generality, to choose the triplet (A, B, C) in "canonical form."

To obtain this representation, we begin by finding the explicit solutions of the mKdV given by (3.12) and (3.22) following an "algorithmic" procedure. Starting

with a real matrix triplet $(A,\, B,\, C)$ which realizes a minimal representation of the function $\Omega(y;\, t)$ and such that the eigenvalues of A have positive real parts, we consider the following equations:

$$A^\dagger Q + QA = C^\dagger C, \quad AN + NA^\dagger = BB^\dagger, \tag{4.1a}$$

$$AP + PA = BC\,. \tag{4.1b}$$

Equations (4.1a) are the so-called Lyapunov equations, instead (4.1b) is known as a Sylvester equation. These equations are studied in detail in [12] where it is proved that, under our hypotheses on the triplet $(A,\, B,\, C)$, they are uniquely solvable. We have the following

Theorem 4.1 *Suppose that the triplet $(A,\, B,\, C)$ is real and is a minimal representation of the kernel $\Omega(y,t)$, and that the eigenvalues of the matrix A have positive real parts. Then, the unique solutions Q, N of the Lyapunov equations and the unique solution P of the Sylvester equation are such that*

1. *The matrices Q, N and P are real, and Q and N are selfadjoint.*
2. *The matrices Q and N can be expressed via* (3.8)*, instead the matrix P is given by the formula* (3.21)*. Moreover, Q and N are simultaneously invertible, and also the matrix P is invertible.*

A proof of this theorem can be found in [7] and [12].

Suppose that we are able to solve (4.1a) (respectively, (4.1b)). As a consequence of Theorem 4.1, their matrix solutions are those introduced in the preceding section by (3.8) (respectively, (3.21)). For these reasons, knowing the solutions Q and N (respectively, P), we can construct, in a unique way, the matrix $F(x,\, t)$ via eq. (3.11) (respectively, $E(x,\, t)$ given by (3.21)), and, finally, we can write down the solution of the mKdV by (3.12) (respectively, (3.22)).

It is natural to look for a larger class including triplets such that the solutions of the corresponding Lyapunov or Sylvester equations have the same properties as in Theorem 4.1. In fact, for every triplet in this class, we can repeat the procedure above introduced. For this reasons we introduce the following

Definition 4.2 *We say that the triplet (A,B,C) of size p belongs to the admissible class if the following conditions are met:*

(i) The matrices A, B, and C are real.

(ii) The triplet (A,B,C) corresponds to the minimal realization when used in the right-hand side of (3.3).

(iii) None of the eigenvalues of A is purely imaginary and no two eigenvalues of A can occur symmetrically with respect to the imaginary axis in the complex λ-plane.

Definition 4.3 *Two triplets $(\tilde{A},\tilde{B},\tilde{C})$ and (A,B,C) are called equivalent if they lead to the same potential $u(x,t)$.*

Let us now consider a triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ in the admissible class. What can we state about the solutions of the corresponding Lyapunov or Sylvester equations

$$\tilde{A}^\dagger \tilde{Q} + \tilde{Q}\tilde{A} = \tilde{C}^\dagger \tilde{C}, \quad \tilde{A}\tilde{N} + \tilde{N}\tilde{A}^\dagger = \tilde{B}\tilde{B}^\dagger, \tag{4.2a}$$

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A} = \tilde{B}\tilde{C}\,? \tag{4.2b}$$

In [7, 12] the reader will find the proof of the following

Theorem 4.4 *If the triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ belongs to the admissible class, then the following statements hold:*

1) Equations (4.2a) *and* (4.2b) *are uniquely solvable.*

2) The matrix solutions $\tilde{Q}$ and $\tilde{N}$ of (4.2a) *are selfadjoint. Moreover, $\tilde{Q}$ and $\tilde{N}$ are simultaneously invertible and also $\tilde{P}$ is invertible.*

3) The matrices

$$\tilde{F}(x,t) = e^{2\tilde{A}^\dagger x - 8(\tilde{A}^\dagger)^3 t} + \tilde{Q} e^{-2\tilde{A}x + 8\tilde{A}^3 t} \tilde{N}, \tag{4.3a}$$

$$\tilde{E}(x,\, t) = e^{2\tilde{A}x - 8\tilde{A}^3 t} + \tilde{P} e^{-2\tilde{A}x + 8\tilde{A}^3 t} \tilde{P}, \tag{4.3b}$$

are invertible on the entire xt-plane.

4) The functions

$$\tilde{u}(x;t) = -2\tilde{B}^\dagger \tilde{F}^{-1}(x,t)\tilde{C}^\dagger, \quad \tilde{v}(x;t) = -2\tilde{C}\tilde{E}^{-1}(x,t)\tilde{B}\,. \tag{4.4}$$

yield two different but equivalent explicit solutions of (1.1)*. Moreover, they are analytic on the entire xt-plane and tend to zero exponentially for each fixed t as $x \to \pm\infty$.*

Then for every triplet in the admissible class we can apply the algorithmic procedure above.

Now, a natural question is: Starting from a triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ in the admissible class, is it possible to construct an equivalent triplet $(A,\, B,\, C)$ such that the matrices $A,\, B,\, C$ are real and are a minimal representation of the function $\Omega(y,\, t)$, and the eigenvalues of A have positive real parts (i.e., a triplet $(A,\, B,\, C)$ of type to this considered in Section 3)? The answer to this question is affirmative. For the sake of space we refer the reader to [7] or [6] where a complete solution to this problem is presented. In particular, eqs. (4.7) and (4.8) of [7] yield the triplet $(A,\, B,\, C)$ by starting from the triplet $(\tilde{A}, \tilde{B}, \tilde{C})$, while eqs. (4.10), (4.11) in the same paper explain how to construct $Q,\, N,\, E,\, F$ from $\tilde{Q},\, \tilde{N},\, \tilde{E},\, \tilde{F}$. As a consequence, the following theorem allows us to understand which is the "canonical way" to choose the triplet $(A,\, B,\, C)$ in (3.4) and, consequently, in the explicit formulas (3.12) and (3.22).

Theorem 4.5 *To any admissible triplet $(\tilde{A}, \tilde{B}, \tilde{C})$, one can associate a special admissible triplet (A, B, C), where A has the Jordan canonical form with each Jordan block containing a distinct eigenvalue having a positive real part, the column B consists of zeros and ones, and C has real entries. More specifically, for some appropriate positive integer m we have*

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix}, \qquad C = \begin{pmatrix} C_1 & C_2 & \cdots & C_m \end{pmatrix}, \tag{4.5}$$

where in the case of a real (positive) eigenvalue ω_j of A_j the corresponding blocks are given by

$$C_j := \begin{pmatrix} c_{jn_j} & \cdots & c_{j2} & c_{j1} \end{pmatrix}, \tag{4.6}$$

$$A_j := \begin{pmatrix} \omega_j & -1 & 0 & \cdots & 0 & 0 \\ 0 & \omega_j & -1 & \cdots & 0 & 0 \\ 0 & 0 & \omega_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_j & -1 \\ 0 & 0 & 0 & \cdots & 0 & \omega_j \end{pmatrix}, \qquad B_j := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{4.7}$$

with A_j having size $n_j \times n_j$, B_j size $n_j \times 1$, C_j size $1 \times n_j$, and the constant c_{jn_j} is nonzero. In the case of complex eigenvalues, which must appear in pairs as $\alpha_j \pm i\beta_j$ with $\alpha_j > 0$, the corresponding blocks are given by

$$C_j := \begin{pmatrix} \gamma_{jn_j} & \epsilon_{jn_j} & \dots & \gamma_{j1} & \epsilon_{j1} \end{pmatrix}, \tag{4.8}$$

$$A_j := \begin{pmatrix} \Lambda_j & -I_2 & 0 & \dots & 0 & 0 \\ 0 & \Lambda_j & -I_2 & \dots & 0 & 0 \\ 0 & 0 & \Lambda_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \Lambda_j & -I_2 \\ 0 & 0 & 0 & \dots & 0 & \Lambda_j \end{pmatrix}, \quad B_j := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{4.9}$$

where γ_{js} and ϵ_{js} for $s = 1, \dots, n_j$ are real constants with $(\gamma_{jn_j}^2 + \epsilon_{jn_j}^2) > 0$, I_2 denotes the 2×2 unit matrix, each column vector B_j has $2n_j$ components, each A_j has size $2n_j \times 2n_j$, and the 2×2 matrix Λ_j is defined as

$$\Lambda_j := \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}. \tag{4.10}$$

Proof. The real triplet (A, B, C) can be chosen as in Section 3 of [6]. ∎

5 Significant examples

Example 1: Choosing the triplet $(A,\, B,\, C)$ as

$$A = \begin{pmatrix} a \end{pmatrix}, \quad B = \begin{pmatrix} 1 \end{pmatrix}, \quad C = \begin{pmatrix} c \end{pmatrix}$$

where $a > 0$ and $0 \neq c \in \mathbb{R}$ and solving the Sylvester eq. (4.1b), we get

$$P = \begin{pmatrix} \frac{c}{2a} \end{pmatrix} .$$

By using eq. (3.22), we obtain

$$v(x,\, t) = \frac{-2c}{e^{2ax-8a^3t} + \frac{c^2}{4a^2} e^{-2ax+8a^3t}} ,$$

which may be called a "single-soliton solution" to (1.1).

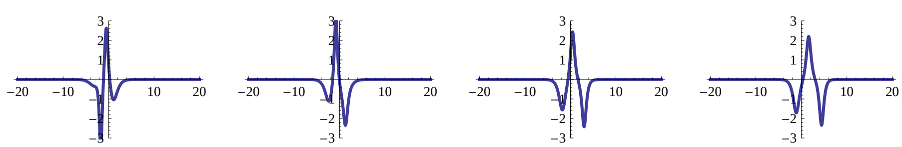
Example 2: In this example we consider the case in which the transmission coefficients have a pole of order three. More precisely, let us consider the following triplet

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 1/2 \end{pmatrix} .$$

It is not difficult to verify that the following matrix P satisfies (4.1b)

$$P = \begin{pmatrix} 1/8 & 7/16 & 5/8 \\ 1/4 & 3/4 & 13/16 \\ 1/2 & 5/4 & 7/8 \end{pmatrix} .$$

In this case it is not a good idea to unzip the solution formula (3.22) in order to write its analytic expression because this representation take a lot of pages! However, using Mathematica it is very easy to plot this solution. In the next figure four different graphs of $u(x,t)$ for four fixed values of t ($t = 0$, $t = 1/4$, $t = 1/2$ and $t = 3/4$) are given.



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